

Analysis of Galerkin and SDFEM on piecewise equidistant meshes for turning point problems exhibiting an interior layer

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Abstract

We consider singularly perturbed boundary value problems with a simple interior turning point whose solutions exhibit an interior layer. These problems are discretised using higher order finite elements on layer-adapted piecewise equidistant meshes proposed by Sun and Stynes. We also study the streamline-diffusion finite element method (SDFEM) for such problems. For these methods error estimates uniform with respect to ε are proven in the energy norm and in the stronger SDFEM-norm, respectively. Numerical experiments confirm the theoretical findings.

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Key words: singular perturbation, turning point, interior layer, layer-adapted meshes, higher order, stabilised FEM.

1 Introduction

We consider singularly perturbed boundary value problems of the form

$$\begin{aligned} -\varepsilon u''(x) + a(x)u'(x) + c(x)u(x) &= f(x) & \text{in } (-1, 1), \\ u(-1) &= \nu_{-1}, \quad u(1) = \nu_1, \end{aligned} \quad (1.1a)$$

with a small parameter $0 < \varepsilon \ll 1$ and sufficiently smooth data a, c, f satisfying

$$a(x) = -(x - x_0)b(x), \quad b(x) > 0, \quad c(x) \geq 0, \quad c(x_0) > 0 \quad (1.1b)$$

for a point $x_0 \in (-1, 1)$. The simple zero x_0 of a is an attractive simple turning point of the problem. Thus, the solution of (1.1) exhibits an interior layer of “cusp”-type [11] at x_0 .

In literature (see i. e. [2], [5, p. 95], [11, Lemma 2.3]) bounds for such interior layers are well known. We have

$$|u^{(i)}(x)| \leq C \left(1 + \left(\varepsilon^{1/2} + |x - x_0| \right)^{\lambda - i} \right) \quad (1.2)$$

where the parameter λ satisfies $0 < \lambda < \bar{\lambda} := c(x_0)/|a'(x_0)|$. Note that the estimate also holds for $\lambda = \bar{\lambda}$, if $\bar{\lambda}$ is not an integer. Otherwise there is an additional logarithmic factor, see references above. In the following we assume $x_0 = 0$ for convenience.

The quest for uniform error estimates for singularly perturbed problems concerns researchers for many years. One of the common strategies is the use of layer-adapted meshes to treat the occurrent boundary and interior layers. Particularly meshes for layers of exponential type have been examined, see e.g. [4] where various problems, numerical methods, and meshes are presented. For this a popular example are, due to their simplicity, the piecewise equidistant Shishkin meshes [9, 7] which are fine only in the layer region. Unfortunately, the layers of “cusp”-type (1.2) do not fade away that quickly and, thus, local refinements do not suffice to capture the layer. Therefore, Sun and Stynes [11, Section 5.1] generalise the standard Shishkin approach and propose a mesh which is equidistant in each of $\mathcal{O}(\ln N)$ subintervals to analyse linear finite elements. Moreover, in [5]

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Liseikin uses graded meshes adapted to (1.2) to prove the ε -uniform first order convergence of an upwind scheme.

For problems of the form (1.1) the finite element method is analysed in [1] on the graded meshes of Liseikin. Using related techniques we shall extend [11] by studying finite elements of order $k \geq 1$ on piecewise uniform meshes with slightly modified parameters, see Section 3. In particular, we prove ε -uniform error estimates of the form $(N^{-1} \ln N)^k$ in a weighted energy norm.

In numerical experiments non-physical oscillations in the error can be observed. In order to damp such behaviour various stabilisation techniques have been proposed in recent years. We shall study the streamline-diffusion finite element method (SDFEM) first introduced by Hughes and Brooks [3]. In Section 4 we prove an error estimate in the SDFEM-norm. For linear elements also a supercloseness result is given which allows to improve the bound for the L^2 -norm error. As an example for the analysis in the context of Shishkin meshes we may refer to Stynes and Tobiska [10] who studied a two-dimensional convection-diffusion problem with exponential boundary layers for Q_p -elements.

Some numerical results are given to illustrate the theoretical findings in Section 5.

Notation: In this paper let C denote a generic constant independent of ε and the number of mesh points. Furthermore, for an interval I we use the usual Sobolev spaces $H^1(I)$, $H_0^1(I)$, $W^{k,\infty}(I)$, and $L^2(I)$. The space of continuous functions on I is written as $C(I)$. We denote by $(\cdot, \cdot)_I$ the usual $L^2(I)$ inner product and by $\|\cdot\|_I$ the $L^2(I)$ -norm. Moreover, the supremum norm on I is written as $\|\cdot\|_{\infty,I}$ and the semi-norm in $H^1(I)$ as $|\cdot|_{1,I}$. If $I = (-1, 1)$, the index I in inner products, norms, and semi-norms will be omitted. Further notation will be introduced later at the beginning of the sections where it is needed.

2 FEM-analysis on arbitrary meshes

The following section is based on the paper of Sun and Stynes [11]. But while their approach merely allows to analyse linear finite elements, the subsequent results give also access to the analysis of finite elements of higher order. We will only consider homogeneous Dirichlet boundary conditions $\nu_{-1} = \nu_1 = 0$. This is no restriction since it can be easily ensured by modifying the right-hand side f . Without loss of generality (cf. [11, Lemma 2.1]) we may assume that

$$(c - \tfrac{1}{2}a')(x) \geq \gamma > 0 \quad \text{for all } x \in [-1, 1], \quad \varepsilon \text{ sufficiently small.} \quad (2.1)$$

For $v, w \in H_0^1((-1, 1))$ we set

$$B_\varepsilon(v, w) := (\varepsilon v', w') + (av', w) + (cv, w).$$

Thanks to (2.1) the bilinear form $B_\varepsilon(\cdot, \cdot)$ is uniformly coercive over $H_0^1((-1, 1)) \times H_0^1((-1, 1))$ in terms of the weighted energy norm $\|\cdot\|_\varepsilon$ defined by

$$\|v\|_\varepsilon := \left(\varepsilon |v|_1^2 + \|v\|^2 \right)^{1/2}.$$

The weak formulation of (1.1) with $\nu_{-1} = \nu_1 = 0$ reads as follows:

Find $u \in H_0^1((-1, 1))$ such that

$$B_\varepsilon(u, v) = (f, v), \quad \text{for all } v \in H_0^1((-1, 1)). \quad (2.2)$$

Let $-1 = x_{-N} < \dots < x_i < \dots < x_N = 1$ define an arbitrary mesh on the interval $[-1, 1]$. The mesh interval length are given by $h_i := x_i - x_{i-1}$. For $k \geq 1$ we denote by $P_k((x_a, x_b))$ the space of polynomial functions of maximal order k over (x_a, x_b) . Furthermore, we define the trial and test space V^N by

$$V^N := \{v \in C([-1, 1]) : v|_{(x_{i-1}, x_i)} \in P_k((x_{i-1}, x_i)) \forall i, v(-1) = v(1) = 0\}.$$

The discrete problem is given by:

Find $u_N \in V^N$ such that

$$B_\varepsilon(u_N, v_N) = (f, v_N), \quad \text{for all } v_N \in V^N. \quad (2.3)$$

Let u_I denote the standard Lagrangian interpolation into V^N , using the mesh points and $k-1$ (arbitrary) inner interpolation point per interval. For example uniform or Gauß-Lobatto points could be chosen.

Assuming $u \in W^{k+1,\infty}((x_{i-1}, x_i))$, the standard interpolation theory leads to the error estimates: For all $j = 0, \dots, k+1$

$$\|(u - u_I)^{(j)}\|_{\infty, (x_{i-1}, x_i)} \leq Ch_i^{k+1-j} \|u^{(k+1)}\|_{\infty, (x_{i-1}, x_i)} \quad (2.4)$$

and

$$\|(u - u_I)^{(j)}\|_{\infty, (x_{i-1}, x_i)} \leq C \|u^{(j)}\|_{\infty, (x_{i-1}, x_i)} \quad (2.5)$$

where C depends on the choice of the inner interpolation points. Furthermore, for all $v_N \in V^N$ the inverse inequality

$$\|v'_N\|_{\infty, (x_{i-1}, x_i)} \leq Ch_i^{-1} \|v_N\|_{\infty, (x_{i-1}, x_i)} \quad (2.6)$$

holds.

In order to estimate the error of the finite element solution we use the splitting

$$u - u_N = (u - u_I) + (u_I - u_N). \quad (2.7)$$

The next lemma shows that the energy norm of the second term can be estimated by knowing some interpolation error bounds only. The given approach works for finite elements of arbitrary order $k \geq 1$.

Lemma 2.1

Let u be the solution of (1.1) and u_N the solution of (2.3) on an arbitrary mesh. Then we have

$$\|u_I - u_N\|_\varepsilon \leq C \left(\|u_I - u\|_\varepsilon + \|x(u_I - u)'\| \right).$$

Proof: Using the coercivity of $B_\varepsilon(\cdot, \cdot)$ and orthogonality, we obtain

$$\min\{\gamma, 1\} \|u_I - u_N\|_\varepsilon^2 \leq B_\varepsilon(u_I - u_N, u_I - u_N) = B_\varepsilon(u_I - u, u_I - u_N). \quad (2.8)$$

Now, Cauchy Schwarz' inequality yields

$$\begin{aligned} B_\varepsilon(u_I - u, u_I - u_N) &= \varepsilon((u_I - u)', (u_I - u_N)') + (a(u_I - u)', u_I - u_N) + (c(u_I - u), u_I - u_N) \\ &\leq \sqrt{\varepsilon} |u_I - u|_1 \sqrt{\varepsilon} |u_I - u_N|_1 + \|a(u_I - u)'\| \|u_I - u_N\| + \|c\|_\infty \|u_I - u\| \|u_I - u_N\|. \end{aligned}$$

Additionally, using the fact that $a(x) = -xb(x)$, we get

$$\begin{aligned} B_\varepsilon(u_I - u, u_I - u_N) &\leq \left(\sqrt{\varepsilon} |u_I - u|_1 + \|b\|_\infty \|x(u_I - u)'\| + \|c\|_\infty \|u_I - u\| \right) \|u_I - u_N\|_\varepsilon \\ &\leq C \left(\|u_I - u\|_\varepsilon + \|x(u_I - u)'\| \right) \|u_I - u_N\|_\varepsilon. \end{aligned}$$

Combining this and (2.8) completes the proof. \square

Remark 2.2

For linear finite elements Sun and Stynes [11, Lemma 5.2] proved an estimate of the form

$$\|u_I - u_N\|_\varepsilon \leq C \left(\|u - u_I\|^{1/2} + \max_i h_i^2 \right),$$

see also [1, Lemma 3.7]. Aside from the fact that their argumentation works for linear elements only, such an estimate would not enable optimal estimates for finite elements of higher order. \clubsuit

Remark 2.3

In the setting of Lemma 2.1 we also have

$$\|u_I - u_N\|_\varepsilon \leq C \|u_I - u\|_\epsilon + C \left(\sum_{i=-N+1}^N h_i^{-2} \|x(u_I - u)\|_{(x_{i-1}, x_i)}^2 \right)^{1/2}$$

which is proven in [1, Lemma 3.1]. ♣

3 The piecewise equidistant meshes of Sun and Stynes

This section is devoted to the study of the piecewise equidistant meshes proposed by Sun and Stynes in [11, Section 5.1]. They generalise the standard approach of Shishkin and introduce a mesh that is equidistant on each of $\mathcal{O}(\ln N)$ subintervals. Because of symmetry, the mesh will be described for $x \geq 0$ only. In order to enable the analysis of finite elements of order $k \geq 1$, we slightly modify the mesh parameters.

In the following we assume that λ from (1.2) lies in $(0, k+1)$ which is the most difficult case. Otherwise all crucial derivatives of the solution could be bounded by a generic constant independent of ε and consequently optimal order ε -uniform estimates could be proven with standard methods on uniform meshes.

For $\varepsilon \in (0, 1]$ and given positive integer N we set

$$\sigma = \max \left\{ \varepsilon^{(1-\lambda/(k+1))/2}, N^{-(2k+1)} \right\} \quad (3.1)$$

and

$$\mathcal{K} = \left\lfloor 1 - \frac{\ln(\sigma)}{\ln(10)} \right\rfloor, \quad (3.2)$$

where $\lfloor z \rfloor$ denotes the largest integer less or equal to z .

The piecewise equidistant mesh is constructed as follows: The interval $(0, 1]$ is partitioned into the $\mathcal{K} + 1$ subintervals $(0, 10^{-\mathcal{K}}]$, $(10^{-\mathcal{K}}, 10^{-\mathcal{K}+1}]$, \dots , $(10^{-1}, 1]$. Then in a second step each of these subintervals is divided uniformly into $\lfloor N/(\mathcal{K} + 1) \rfloor$ parts. For simplicity we assume that $\lfloor N/(\mathcal{K} + 1) \rfloor = N/(\mathcal{K} + 1)$. Hence, by construction we have

$$h_i = (\mathcal{K} + 1)10^{-\mathcal{K}}N^{-1}, \quad \text{for } x_i \in (0, 10^{-\mathcal{K}}] \quad (3.3)$$

and

$$h_i = 9(\mathcal{K} + 1)10^{-l}N^{-1}, \quad \text{for } x_i \in (10^{-l}, 10^{-l+1}] \quad \text{and } l = 1, \dots, \mathcal{K}. \quad (3.4)$$

From (3.1), (3.2), and the properties of the logarithm we see that

$$\mathcal{K} + 1 \leq 2 - \frac{\ln(\sigma)}{\ln(10)} \leq 2 + \min \left\{ \frac{1 - \lambda/(k+1)}{2} \frac{|\ln(\varepsilon)|}{\ln(10)}, (2k+1) \frac{\ln(N)}{\ln(10)} \right\}.$$

For N sufficiently large (dependent on k) this estimate yields $\mathcal{K} + 1 \leq N$. Furthermore, we obtain

$$\mathcal{K} + 1 \leq C \ln N. \quad (3.5)$$

From (3.2) we have $\ln(10) - \ln(\sigma) \geq \mathcal{K} \ln(10) > -\ln(\sigma)$ which implies

$$10^{-1}\sigma \leq 10^{-\mathcal{K}} < \sigma. \quad (3.6)$$

The next lemma is a generalisation of [11, Lemma 5.3] and provides some important basic results for the mesh intervals of the piecewise equidistant mesh.

Lemma 3.1

Let $j = 0, 1$. The following inequalities hold

$$h_i^{k+1-j} \left(x_{i-1} + \varepsilon^{1/2} \right)^{\lambda-(k+1-j)} \leq C \left((\mathcal{K} + 1)N^{-1} \right)^{k+1-j}, \quad \text{for } x_i \in (10^{-\mathcal{K}}, 1], \quad (3.7)$$

$$h_i^{k+1-j} \left(x_{i-1} + \varepsilon^{1/2} \right)^{\lambda-(k+1-j)} \leq C (i-1)^{-(k+1-j)}, \quad \text{for } x_i \in (x_1, 10^{-\mathcal{K}}]. \quad (3.8)$$

If $\sigma = \varepsilon^{(1-\lambda/(k+1))/2}$, then

$$h_i^{k+1-j} \left(x_{i-1} + \varepsilon^{1/2} \right)^{\lambda-(k+1-j)} \leq C \left((\mathcal{K} + 1)N^{-1} \right)^{k+1-j}, \quad \text{for } x_i \in (0, 10^{-\mathcal{K}}]. \quad (3.9)$$

In general, the mesh interval length can be bounded by

$$h_i \leq (\mathcal{K} + 1)N^{-1}.$$

Furthermore, in the case of $\sigma = N^{-(2k+1)}$, we have

$$x_1 = h_1 \leq (\mathcal{K} + 1)N^{-2(k+1)}. \quad (3.10)$$

Proof: In order to prove the last estimate, let $\sigma = N^{-(2k+1)}$. Combining (3.3) with (3.6) yields

$$x_1 = h_1 = (\mathcal{K} + 1)10^{-\mathcal{K}}N^{-1} \leq (\mathcal{K} + 1)\sigma N^{-1} = (\mathcal{K} + 1)N^{-2(k+1)}.$$

The general estimates for h_i follow from (3.3) and (3.4).

We assume $\lambda - (k + 1 - j) < 0$ in the following. Otherwise $(x_{i-1} + \varepsilon^{1/2})^{\lambda-(k+1-j)} \leq C$ would allow to deduce the wanted bounds very easily.

So, let $x_i \in (10^{-l}, 10^{-l+1}]$ for some $l \in \{1, \dots, \mathcal{K}\}$. With (3.4) we obtain

$$\begin{aligned} h_i^{k+1-j} \left(x_{i-1} + \varepsilon^{1/2} \right)^{\lambda-(k+1-j)} &\leq (9(\mathcal{K} + 1)10^{-l}N^{-1})^{k+1-j} \left(10^{-l} + \varepsilon^{1/2} \right)^{\lambda-(k+1-j)} \\ &\leq C \left((\mathcal{K} + 1)N^{-1} \right)^{k+1-j} (10^{-l})^{k+1-j} (10^{-l})^{\lambda-(k+1-j)} \\ &= C \left((\mathcal{K} + 1)N^{-1} \right)^{k+1-j} 10^{-\lambda l} \leq C \left((\mathcal{K} + 1)N^{-1} \right)^{k+1-j}. \end{aligned}$$

For $x_i \in (x_1, 10^{-\mathcal{K}}]$ the fact that the mesh is equidistant in this interval implies

$$\begin{aligned} h_i^{k+1-j} \left(x_{i-1} + \varepsilon^{1/2} \right)^{\lambda-(k+1-j)} &= (x_{i-1}(i-1)^{-1})^{k+1-j} \left(x_{i-1} + \varepsilon^{1/2} \right)^{\lambda-(k+1-j)} \\ &\leq (i-1)^{-(k+1-j)} x_{i-1}^{\lambda} \\ &\leq (i-1)^{-(k+1-j)}. \end{aligned}$$

Finally, let $\sigma = \varepsilon^{(1-\lambda/(k+1))/2}$ and $x_i \in (0, 10^{-\mathcal{K}}]$. Using (3.3) and (3.6) we get

$$\begin{aligned} h_i^{k+1-j} \left(x_{i-1} + \varepsilon^{1/2} \right)^{\lambda-(k+1-j)} &\leq \left((\mathcal{K} + 1)\sigma N^{-1} \right)^{k+1-j} \varepsilon^{(\lambda-(k+1-j))/2} \\ &\leq C \left((\mathcal{K} + 1)N^{-1} \right)^{k+1-j} \varepsilon^{\lambda j/(2(k+1))} \\ &\leq C \left((\mathcal{K} + 1)N^{-1} \right)^{k+1-j}. \quad \square \end{aligned}$$

The interpolation error on the layer-adapted piecewise equidistant mesh proposed by Sun and Stynes shall be bounded in the following lemma which is a generalisation of [11, Lemma 5.4]. We also refer to [1, Lemma 3.3, Lemma 3.4] where a similar argumentation is used to estimate the interpolation error on special graded meshes.

Lemma 3.2

Let u be the solution of problem (1.1) and $u_I \in V^N$ be its interpolant on the piecewise equidistant mesh given by (3.1) – (3.4). Then

$$\|u - u_I\| \leq C \left((\mathcal{K} + 1)N^{-1} \right)^{k+1} \quad (3.11)$$

and

$$\|u - u_I\|_{\varepsilon} + \|x(u - u_I)'\| \leq C \left((\mathcal{K} + 1)N^{-1} \right)^k. \quad (3.12)$$

Proof: Thanks to the symmetry of the problem, we shall consider only $x \in [0, 1]$. Furthermore, we use $j \in \{0, 1\}$ to switch between the L^2 -norm term and the ε -weighted H^1 -seminorm term. The estimate for $\|x(u - u_I)'\|$ is also covered by $j = 1$.

Let $x \in (x_{i-1}, x_i)$ for some i , where $x_i \in (10^{-\mathcal{K}}, 1]$. Then

$$\begin{aligned} (\varepsilon^{j/2} + jx) \left| (u - u_I)^{(j)}(x) \right| &\leq C(\varepsilon^{j/2} + jx) h_i^{k+1-j} \|u^{(k+1)}\|_{\infty, (x_{i-1}, x_i)} \\ &\leq C(\varepsilon^{j/2} + j(x_{i-1} + h_i)) h_i^{k+1-j} \left(1 + (x_{i-1} + \varepsilon^{1/2})^{\lambda-(k+1)} \right) \\ &\leq C((\mathcal{K} + 1)N^{-1})^{k+1-j}, \end{aligned}$$

where we used (2.4), (1.2), and Lemma 3.1. Hence,

$$\varepsilon^j \int_{10^{-\mathcal{K}}}^1 \left((u - u_I)^{(j)}(x) \right)^2 dx + j^2 \int_{10^{-\mathcal{K}}}^1 \left(x(u - u_I)^{(j)}(x) \right)^2 dx \leq C((\mathcal{K} + 1)N^{-1})^{2(k+1-j)}.$$

Now, let $x \in (x_{i-1}, x_i)$, where $x_i \in (0, 10^{-\mathcal{K}}]$. We consider two different cases for σ .

First, if $\sigma = \varepsilon^{(1-\lambda/(k+1))/2}$ then as above Lemma 3.1 yields

$$\begin{aligned} (\varepsilon^{j/2} + jx) \left| (u - u_I)^{(j)}(x) \right| &\leq C(\varepsilon^{j/2} + j(x_{i-1} + h_i)) h_i^{k+1-j} \left(1 + (x_{i-1} + \varepsilon^{1/2})^{\lambda-(k+1)} \right) \\ &\leq C((\mathcal{K} + 1)N^{-1})^{k+1-j} \end{aligned}$$

and therefore

$$\varepsilon^j \int_0^{10^{-\mathcal{K}}} \left((u - u_I)^{(j)}(x) \right)^2 dx + j^2 \int_0^{10^{-\mathcal{K}}} \left(x(u - u_I)^{(j)}(x) \right)^2 dx \leq C((\mathcal{K} + 1)N^{-1})^{2(k+1-j)}.$$

Finally, let $\sigma = N^{-(2k+1)}$. The integral over $(0, x_1)$ is estimated directly. By (2.5), (1.2), and Lemma 3.1 especially (3.10), we have

$$\begin{aligned} \varepsilon^j \int_0^{x_1} \left((u - u_I)^{(j)}(x) \right)^2 dx &\leq C\varepsilon^j \int_0^{x_1} \|u^{(j)}\|_{\infty, (0, x_1)}^2 dx \\ &\leq C\varepsilon^j \left(1 + \varepsilon^{(\lambda-j)/2} \right)^2 x_1 \\ &\leq C(\mathcal{K} + 1)N^{-2(k+1)} \end{aligned}$$

and

$$\begin{aligned} \int_0^{x_1} \left(x(u - u_I)'(x) \right)^2 dx &\leq 2 \|xu'\|_{\infty, (0, x_1)}^2 \int_0^{x_1} 1 dx + 2 \|u_I'\|_{\infty, (0, x_1)}^2 \int_0^{x_1} x^2 dx \\ &\leq 2 \|xu'\|_{\infty, (0, x_1)}^2 x_1 + Ch_1^{-2} \|u_I\|_{\infty, (0, x_1)}^2 x_1^3 \\ &\leq C(\mathcal{K} + 1)N^{-2(k+1)}, \end{aligned}$$

where additionally the inverse inequality (2.6) and the stability $\|u_I\|_{\infty} \leq C \|u\|_{\infty}$ is applied. Now, let $x \in (x_{i-1}, x_i) \subseteq (x_1, 10^{-\mathcal{K}}]$. Then

$$\begin{aligned} (\varepsilon^{j/2} + jx) \left| (u - u_I)^{(j)}(x) \right| &\leq C(\varepsilon^{j/2} + j(x_{i-1} + h_i)) h_i^{k+1-j} \left(1 + (x_{i-1} + \varepsilon^{1/2})^{\lambda-(k+1)} \right) \\ &\leq C \left((\varepsilon^{j/2} + j10^{-\mathcal{K}}) h_i + (i-1)^{-(k+1-j)} \right), \end{aligned}$$

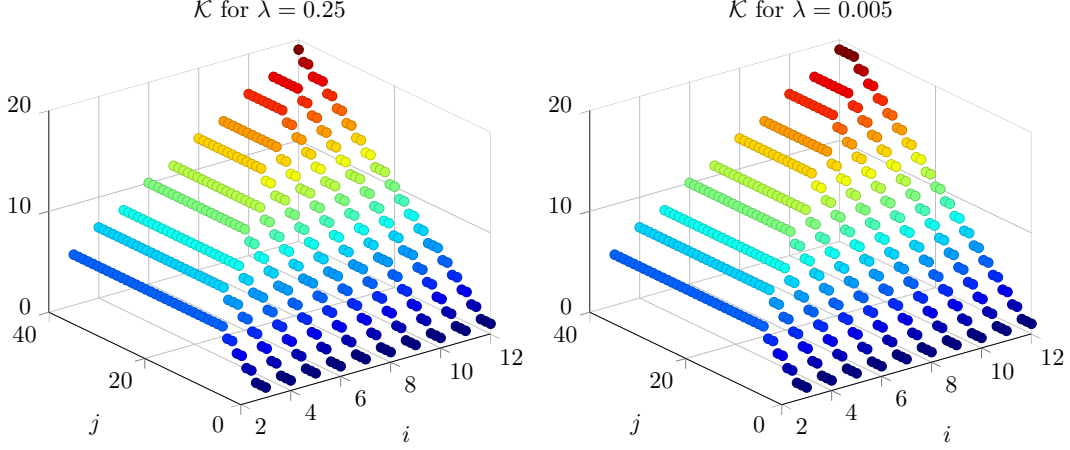


Figure 1: \mathcal{K} for $N = 2^i$, $\varepsilon = 10^{-j}$, $k = 2$, and $\lambda = 0.25$ (left) and $\lambda = 0.005$ (right).

for $i = 2, \dots, N/(\mathcal{K} + 1)$, where Lemma 3.1 especially (3.8) is used. Hence, with (3.10)

$$\begin{aligned}
& \varepsilon^j \int_{x_1}^{10^{-\mathcal{K}}} \left((u - u_I)^{(j)}(x) \right)^2 dx + j^2 \int_{x_1}^{10^{-\mathcal{K}}} \left(x(u - u_I)^{(j)}(x) \right)^2 dx \\
& \leq C \sum_{i=2}^{N/(\mathcal{K}+1)} h_i \left((\varepsilon^{j/2} + j10^{-\mathcal{K}}) h_i + (i-1)^{-(k+1-j)} \right)^2 \\
& \leq C x_1 \sum_{i=2}^{N/(\mathcal{K}+1)} \left(h_i + (i-1)^{-2} \right) \\
& \leq C x_1 \left(10^{-\mathcal{K}} + \frac{\pi^2}{6} \right) \\
& \leq C(\mathcal{K} + 1) N^{-2(k+1)}.
\end{aligned}$$

Combining the above estimates for $j = 0$ and using symmetry on $[-1, 0]$ we get (3.11). This estimate together with the above estimates for $j = 1$ immediately gives (3.12). \square

Now, we are able to prove the ε -uniform error estimate in the energy norm for finite elements of order $k \geq 1$ on the piecewise equidistant mesh of Sun and Stynes.

Theorem 3.3

Let u be the solution of (1.1) and u_N the solution of (2.3) on the piecewise equidistant mesh given by (3.1) – (3.4). Then we have

$$\|u - u_N\|_{\varepsilon} \leq C((\mathcal{K} + 1)N^{-1})^k \leq C(N^{-1} \ln N)^k.$$

Proof: The bound in the energy norm follows easily from the triangle inequality, Lemma 2.1 and (3.12). The second inequality is an immediately consequence of (3.5). \square

Remark 3.4

In the numerically studied ranges ($\varepsilon \in [10^{-14}, 1]$ and $\lambda \in \{0.005, 0.25\}$) the ε -dependent term in (3.1) is dominant and thus \mathcal{K} constant in N , cf. also Figure 1. Therefore, the logarithmic factor in the estimate of Theorem 3.3 could not be seen in the numerical experiments.

4 SDFEM-analysis on arbitrary and piecewise equidistant meshes

In this section the streamline-diffusion finite element method (SDFEM) is studied. For convenience we use the shorter notation $I_i := (x_{i-1}, x_i)$ for all $i = -N + 1, \dots, N$. In order to increase the

stability some extra terms are added to the weak formulation. We set

$$B_{\text{SD}}(v, w) := B_\varepsilon(v, w) + B_{\text{Stab}}(v, w)$$

where

$$B_{\text{Stab}}(v, w) := \sum_{i=-N+1}^N \delta_i \int_{x_{i-1}}^{x_i} (-\varepsilon u'' + au' + cu)(x)(av')(x)dx,$$

and

$$f_{\text{Stab}}(v) := \sum_{i=-N+1}^N \delta_i \int_{x_{i-1}}^{x_i} f(x)(av')(x)dx,$$

with stabilisation parameters $\delta_i \geq 0$ to be defined later. Now, the discrete problem is given by:

Find $u_N \in V^N$ such that

$$B_{\text{SD}}(u_N, v_N) = (f, v_N) + f_{\text{Stab}}(v_N), \quad \text{for all } v_N \in V^N. \quad (4.1)$$

Note that the method is consistent, i.e. for $u \in H^2((-1, 1))$ of (2.2) we have

$$B_{\text{SD}}(u, v_N) = (f, v_N) + f_{\text{Stab}}(v_N), \quad \text{for all } v_N \in V^N.$$

For our analysis we define the SDFEM-norm by

$$\|v\|_{\text{SD}} := \left(\varepsilon |v|_1^2 + \|v\|^2 + \sum_{i=N-1}^N \|\sqrt{\delta_i} av'\|_{I_i}^2 \right)^{1/2}.$$

Because of the additional terms this norm is stronger than the energy norm.

It holds the following inverse inequality

$$\|v_N''\|_{I_i} \leq c_{\text{inv}} h_i^{-1} \|v_N'\|_{I_i} \quad \forall v_N \in V^N.$$

Thus, imposing the requirement

$$0 \leq \delta_i \leq \frac{1}{2} \min \left\{ \frac{h_i^2}{\varepsilon c_{\text{inv}}}, \frac{\gamma}{\|c\|_\infty^2} \right\} \quad (4.2a)$$

or the assumption

$$0 \leq \delta_i \leq \frac{\gamma}{\|c\|_\infty^2} \quad (4.2b)$$

for linear finite elements, respectively, we obtain analogously to [8, p. 86]

$$B_{\text{SD}}(v_N, v_N) \geq \frac{1}{2} \|v_N\|_{\text{SD}}^2 \quad \forall v_N \in V^N. \quad (4.3)$$

4.1 Higher order finite elements

Using the splitting (2.7) our analysis starts with the following lemma.

Lemma 4.1

Let u be the solution of (1.1), u_N the solution of (4.1), and u_I the interpolant of u on an arbitrary mesh. Furthermore, choose δ_i such that (4.2) is satisfied. Then we have

$$\begin{aligned} \|u_I - u_N\|_{\text{SD}} &\leq C \left(1 + \max_i \sqrt{\delta_i} \right) \left(\|u_I - u\|_\varepsilon + \|x(u_I - u)'\| \right) \\ &\quad + C \left(\sum_{i=-N+1}^N \min \left\{ \|\sqrt{\delta_i} \varepsilon (u_I - u)''\|_{I_i}^2, \|\delta_i \sqrt{\varepsilon} x(u_I - u)''\|_{I_i}^2 \right\} \right)^{1/2}. \end{aligned}$$

Proof: By (4.3) and due to orthogonality which is implied by consistency, we have

$$\begin{aligned} \frac{1}{2} \|u_I - u_N\|_{\text{SD}}^2 &\leq B_{\text{SD}}(u_I - u_N, u_I - u_N) = B_{\text{SD}}(u_I - u, u_I - u_N) \\ &= B_\varepsilon(u_I - u, u_I - u_N) + B_{\text{Stab}}(u_I - u, u_I - u_N). \end{aligned}$$

As in the proof of Lemma 2.1 we obtain for the first term

$$B_\varepsilon(u_I - u, u_I - u_N) \leq C \left(\|u_I - u\|_\varepsilon + \|x(u_I - u)'\| \right) \|u_I - u_N\|_\varepsilon.$$

It remains to estimate the second term

$$\begin{aligned} B_{\text{Stab}}(u_I - u, u_I - u_N) \\ = \sum_i \delta_i \int_{I_i} (-\varepsilon(u_I - u)'' + a(u_I - u)' + c(u_I - u))(x)(a(u_I - u_N)')(x) dx. \end{aligned} \quad (4.4)$$

Applying Cauchy Schwarz' inequality and using the fact that $a(x) = -xb(x)$ we gain for the last two summands in (4.4)

$$\begin{aligned} &\left| \sum_i \delta_i \int_{I_i} (a(u_I - u)' + c(u_I - u))(x)(a(u_I - u_N)')(x) dx \right| \\ &\leq \sum_i \sqrt{\delta_i} (\|a(u_I - u)'\|_{I_i} + C \|u_I - u\|_{I_i}) \|\sqrt{\delta_i} a(u_I - u_N)'\|_{I_i} \\ &\leq C \max_i \sqrt{\delta_i} (\|x(u_I - u)'\| + \|u_I - u\|) \left(\sum_i \|\sqrt{\delta_i} a(u_I - u_N)'\|_{I_i}^2 \right)^{1/2}. \end{aligned} \quad (4.5)$$

Furthermore, we have on the one hand

$$\left| \delta_i \int_{I_i} \varepsilon(u_I - u)'' a(u_I - u_N)' dx \right| \leq \|\sqrt{\delta_i} \varepsilon(u_I - u)''\|_{I_i} \|\sqrt{\delta_i} a(u_I - u_N)'\|_{I_i}$$

and on the other hand by the properties of a

$$\begin{aligned} \left| \delta_i \int_{I_i} \varepsilon(u_I - u)'' a(u_I - u_N)' dx \right| &\leq \|\delta_i \sqrt{\varepsilon} a(u_I - u)''\|_{I_i} \sqrt{\varepsilon} |u_I - u_N|_{1, I_i} \\ &\leq C \|\delta_i \sqrt{\varepsilon} x(u_I - u)''\|_{I_i} \sqrt{\varepsilon} |u_I - u_N|_{1, I_i}. \end{aligned}$$

Thus, for the first term in (4.4) Cauchy Schwarz' inequality yields

$$\begin{aligned} &\left| \sum_i \delta_i \int_{I_i} \varepsilon(u_I - u)'' a(u_I - u_N)' dx \right| \\ &\leq C \left(\sum_i \min \left\{ \|\sqrt{\delta_i} \varepsilon(u_I - u)''\|_{I_i}^2, \|\delta_i \sqrt{\varepsilon} x(u_I - u)''\|_{I_i}^2 \right\} \right)^{1/2} \|u_I - u_N\|_{\text{SD}}. \end{aligned}$$

Combining the above estimates we are done. \square

In Section 3 several interpolation error terms have been already studied and bounded. It remains to estimate the last term in Lemma 4.1.

Lemma 4.2

Let u be the solution of problem (1.1) and $u_I \in V^N$ be its interpolant on the piecewise equidistant mesh given by (3.1) – (3.4). Suppose that

$$0 \leq \delta_i \leq C \min\{1, h_i^2/\varepsilon\}. \quad (4.6)$$

Then

$$\left(\sum_{i=-N+1}^N \min \left\{ \|\sqrt{\delta_i} \varepsilon(u_I - u)''\|_{I_i}^2, \|\delta_i \sqrt{\varepsilon} x(u_I - u)''\|_{I_i}^2 \right\} \right)^{1/2} \leq C \max_i \sqrt{\delta_i} ((\mathcal{K} + 1)N^{-1})^k. \quad (4.7)$$

Proof: The proof is similarly to the proof of Lemma 3.2 but differs in some details.

Let $x \in (x_{i-1}, x_i)$ for some i , where $x_i \in (10^{-\mathcal{K}}, 1]$. Then with $\sqrt{\delta_i \varepsilon} \leq Ch_i$

$$\begin{aligned} \delta_i \sqrt{\varepsilon} |x(u_I - u)''(x)| &\leq C(\sqrt{\delta_i} h_i (x_{i-1} + h_i)) h_i^{k-1} \left(1 + (x_{i-1} + \varepsilon^{1/2})^{\lambda - (k+1)}\right) \\ &\leq C\sqrt{\delta_i} ((\mathcal{K} + 1)N^{-1})^k, \end{aligned}$$

where we used (2.4), (1.2), and Lemma 3.1. Hence, for $N/(\mathcal{K} + 1) + 1 \leq i \leq N$

$$\|\delta_i \sqrt{\varepsilon} x(u_I - u)''\|_{I_i}^2 = \int_{I_i} (\delta_i \sqrt{\varepsilon} x(u_I - u)''(x))^2 dx \leq Ch_i \delta_i ((\mathcal{K} + 1)N^{-1})^{2k}.$$

Now, let $x \in (x_{i-1}, x_i)$, where $x_i \in (0, 10^{-\mathcal{K}}]$. We consider two different cases for σ . First, if $\sigma = \varepsilon^{(1-\lambda/(k+1))/2}$ then as above Lemma 3.1 yields

$$\delta_i \sqrt{\varepsilon} |x(u_I - u)''(x)| \leq C\sqrt{\delta_i} ((\mathcal{K} + 1)N^{-1})^k$$

and therefore

$$\|\delta_i \sqrt{\varepsilon} x(u_I - u)''\|_{I_i}^2 \leq Ch_i \delta_i ((\mathcal{K} + 1)N^{-1})^{2k}$$

for $1 \leq i \leq N/(\mathcal{K} + 1)$.

If $\sigma = N^{-(2k+1)}$ the integral over $(0, x_1)$ can be estimated directly. We have

$$\begin{aligned} \int_0^{x_1} (\sqrt{\delta_1} \varepsilon (u_I - u)''(x))^2 dx &\leq \delta_1 \varepsilon^2 \|(u_I - u)''\|_{\infty, (0, x_1)}^2 \int_0^{x_1} 1 dx \\ &\leq C\delta_1 x_1 \varepsilon^2 \|u''\|_{\infty, (0, x_1)}^2 \\ &\leq C\delta_1 x_1 \leq C\delta_1 (\mathcal{K} + 1)N^{-2(k+1)} \end{aligned}$$

by (2.5), (1.2), and Lemma 3.1 especially (3.10). For $x \in (x_{i-1}, x_i) \subseteq (x_1, 10^{-\mathcal{K}}]$ we use (3.8) to obtain

$$\begin{aligned} \delta_i \sqrt{\varepsilon} |x(u_I - u)''(x)| &\leq C(\sqrt{\delta_i} h_i (x_{i-1} + h_i)) h_i^{k-1} \left(1 + (x_{i-1} + \varepsilon^{1/2})^{\lambda - (k+1)}\right) \\ &\leq C\sqrt{\delta_i} (10^{-\mathcal{K}} h_i^k + (i-1)^{-k}), \quad \text{for } i = 2, \dots, N/(\mathcal{K} + 1). \end{aligned}$$

Thus, the inequality (3.10) yields for $2 \leq i \leq N/(\mathcal{K} + 1)$

$$\begin{aligned} \|\delta_i \sqrt{\varepsilon} x(u_I - u)''\|_{I_i}^2 &\leq C\delta_i h_i (10^{-\mathcal{K}} h_i^k + (i-1)^{-k})^2 \\ &\leq C\delta_i x_1 (h_i + (i-1)^{-2}) \\ &\leq C\delta_i (h_i + (i-1)^{-2}) (\mathcal{K} + 1)N^{-2(k+1)}. \end{aligned}$$

Summing up the above estimates gives in the worst case

$$\begin{aligned} &\sum_{i=1}^N \min \left\{ \|\sqrt{\delta_i} \varepsilon (u_I - u)''\|_{I_i}^2, \|\delta_i \sqrt{\varepsilon} x(u_I - u)''\|_{I_i}^2 \right\} \\ &\leq C \left(\delta_1 + \sum_{i=2}^{N/(\mathcal{K}+1)} \delta_i (h_i + (i-1)^{-2}) \right) (\mathcal{K} + 1)N^{-2(k+1)} + C \sum_{i=1}^N \delta_i h_i ((\mathcal{K} + 1)N^{-1})^{2k} \\ &\leq C \max_i \delta_i ((\mathcal{K} + 1)N^{-1})^{2k} \left(1 + N^{-2} \left(1 + 10^{-\mathcal{K}} + \frac{\pi^2}{6} \right) \right) \\ &\leq C \max_i \delta_i ((\mathcal{K} + 1)N^{-1})^{2k}. \end{aligned}$$

Thanks to symmetry the sum for $i = -N + 1, \dots, 0$ can be bounded analogously and the proof is completed. \square

The previous estimates enable us to prove an error estimate in the SDFEM-norm.

Theorem 4.3

Let u be the solution of (1.1) and u_N the solution of (4.1) on the piecewise equidistant mesh given by (3.1) – (3.4). Furthermore, choose δ_i such that (4.2) and (4.6) are satisfied. Then we have

$$\|u - u_N\|_{\text{SD}} \leq C ((\mathcal{K} + 1)N^{-1})^k \leq C (N^{-1} \ln N)^k.$$

Proof: The fact that $a(x) = -xb(x)$ and $\delta_i \leq C$ imply for all $v \in H^1((-1, 1))$

$$\left(\sum_{i=N-1}^N \|\sqrt{\delta_i}av'\|_{I_i}^2 \right)^{1/2} \leq C \max_i \sqrt{\delta_i} \|xv'\|.$$

Using this, the wanted estimate follows easily from the triangle inequality, Lemma 4.1, (3.12), and (4.7). The second inequality is an immediately consequence of (3.5). \square

4.2 Some improvements for linear elements

Inspecting the proofs of the last section we see that

$$|B_{\text{Stab}}(u_I - u, u_I - u_N)| \leq C \max_i \sqrt{\delta_i} ((\mathcal{K} + 1)N^{-1})^k \|u_I - u_N\|_{\text{SD}}.$$

For linear elements the intermediate estimate (4.5) can be even improved. Indeed, integrating by parts we get

$$\begin{aligned} \int_{I_i} a(u_I - u)' a(u_I - u_N)' dx &= - \int_{I_i} a^2(u_I - u) \underbrace{(u_I - u_N)''}_{=0} dx - \int_{I_i} 2aa'(u_I - u)(u_I - u_N)' dx \\ &\quad + \underbrace{[a^2(u_I - u)(u_I - u_N)']_{x_{i-1}}^{x_i}}_{=0} \\ &= -2 \int_{I_i} a'(u_I - u)a(u_I - u_N)' dx \end{aligned}$$

and by Cauchy Schwarz' inequality

$$\left| \sum_i \delta_i \int_{I_i} a(u_I - u)' a(u_I - u_N)' dx \right| \leq C \max_i \sqrt{\delta_i} \|u_I - u\| \left(\sum_i \|\sqrt{\delta_i}a(u_I - u_N)'\|_{I_i}^2 \right)^{1/2}.$$

Using the argumentation of [11, Theorem 5.1] one can prove for linear elements

$$|B_\varepsilon(u_I - u, u_I - u_N)| \leq C ((\mathcal{K} + 1)N^{-1})^{3/2} \|u_I - u_N\| \leq C (N^{-1} \ln N)^{3/2} \|u_I - u_N\|.$$

Note that the occurring logarithmic factors originate from an estimate of the form $(\mathcal{K} + 1) \leq C \ln N$.

In summary we get the following result.

Theorem 4.4

Let u be the solution of (1.1), $u_N \in V^N$ ($k = 1$) the solution of (4.1), and $u_I \in V^N$ ($k = 1$) the interpolant of u on the piecewise equidistant mesh given by (3.1) – (3.4). Furthermore, choose δ_i such that

$$0 \leq \delta_i \leq \min \left\{ \frac{\gamma}{\|c\|_\infty^2}, \frac{h_i^2}{\varepsilon}, (\mathcal{K} + 1)N^{-1} \right\}.$$

Then we have

$$\|u - u_N\| + \|u_I - u_N\|_{\text{SD}} \leq C ((\mathcal{K} + 1)N^{-1})^{3/2} \leq C (N^{-1} \ln N)^{3/2}.$$

Proof: Revise the proof of Lemma 2.1 using the improved estimates of this section. Then invoke (4.7) and (3.11) to complete the proof. \square

Remark 4.5

For linear Galerkin FEM in [11, Theorem 5.1] the L^2 -norm estimate

$$\|u - u_N\| \leq C (N^{-1} \ln N)^{3/2}$$

is already proven for the piecewise equidistant mesh. \clubsuit

5 Numerical experiments

In this section we present some numerical results to illustrate the theoretical findings of the previous sections. As in [1] we study a test problem taken from [11] whose solution exhibits typical interior layer behaviour of “cusp”-type.

Example 5.1 (see [11])

We consider the singularly perturbed turning point problem

$$\begin{aligned} -\varepsilon u'' - x(1+x^2)u' + \lambda(1+x^3)u &= f, & \text{for } x \in (-1, 1), \\ u(-1) &= u(1) = 0, \end{aligned}$$

where the right-hand side $f(x)$ is chosen such that the solution $u(x)$ is given by

$$u(x) = (x^2 + \varepsilon)^{\lambda/2} + x(x^2 + \varepsilon)^{(\lambda-1)/2} - (1 + \varepsilon)^{\lambda/2} \left(1 + x(1 + \varepsilon)^{-1/2}\right).$$

Note that the parameter λ in the problem coincide with the quantity $\bar{\lambda} = c(0)/|a'(0)|$.

All computations were performed using a FEM-code based on SOFE by Lars Ludwig [6]. Motivated by our error estimates we calculate the convergence rates by

$$r = (\ln E_{\varepsilon,N} - \ln E_{\varepsilon,2N}) / \ln 2$$

for given errors $E_{\varepsilon,N}$. In order to ensure that the used meshes consists of exactly $2N$ mesh intervals, which is presumed in this formula, we adjust the mesh like in [11, Section 6]:

Let $N_0 = N - (\mathcal{K} + 1)n_0$ where $n_0 = \lfloor N/(\mathcal{K} + 1) \rfloor$. Then we uniformly partition each of the subintervals $(0, 10^{-\mathcal{K}}], \dots, (10^{-N_0-1}, 10^{-N_0}]$ by n_0 points and each of the remaining subintervals $(10^{-N_0}, 10^{-N_0+1}], \dots, (10^{-1}, 1]$ by $n_0 + 1$ points. The arising mesh is still piecewise equidistant and has exactly $2N$ mesh intervals.

For the streamline-diffusion finite element method we choose the stabilisation parameter as

$$\delta_i = C_0 \min \{h_i^2/\varepsilon, h_i\}$$

which is the standard choice, see e.g. [8, p. 87]. Although it is not necessary to have $\delta_i \leq C_0 h_i$ by theory, numerical tests suggest to favour this definition. We use $C_0 = 1$ for computations.

We plot the energy norm error for P_k -FEM and the SDFEM-norm error for P_k -SDFEM, $k = 1, \dots, 4$ in Figure 2 where the methods were applied to Example 5.1 with $\varepsilon = 10^{-10}$ and $\lambda = 0.005$. The expected orders of convergence, cf. Theorem 3.3, Theorem 4.4, can be clearly seen. The magnitude of the errors is similar for both methods. Recall that the SDFEM-norm is stronger than the energy norm. The numerical results also suggest that the errors are uniform with respect to ε . They stay stable for small ε , see Table 1 for FEM and Table 2 for SDFEM.

In Figure 3 and Figure 4 the errors of both methods for P_2 -elements applied to Example 5.1 with $\varepsilon = 10^{-6}$, $\lambda = 0.25$ and $\lambda = 0.005$, respectively, on a mesh with $N = 128$ are plotted. While for the finite element method the error is clearly oscillating in $(0, 1)$, this behaviour is damped and even prevented in a wide range of the interval when the stabilisation technique is used.

For linear SDFEM applied to Example 5.1 with $\varepsilon = 10^{-10}$ and $\lambda = 0.005$ the errors in SDFEM-norm, energy norm, and L^2 -norm can be compared in Table 3. The calculated convergence rates coincide with the rates expected from standard theory for non singularly perturbed problems. So

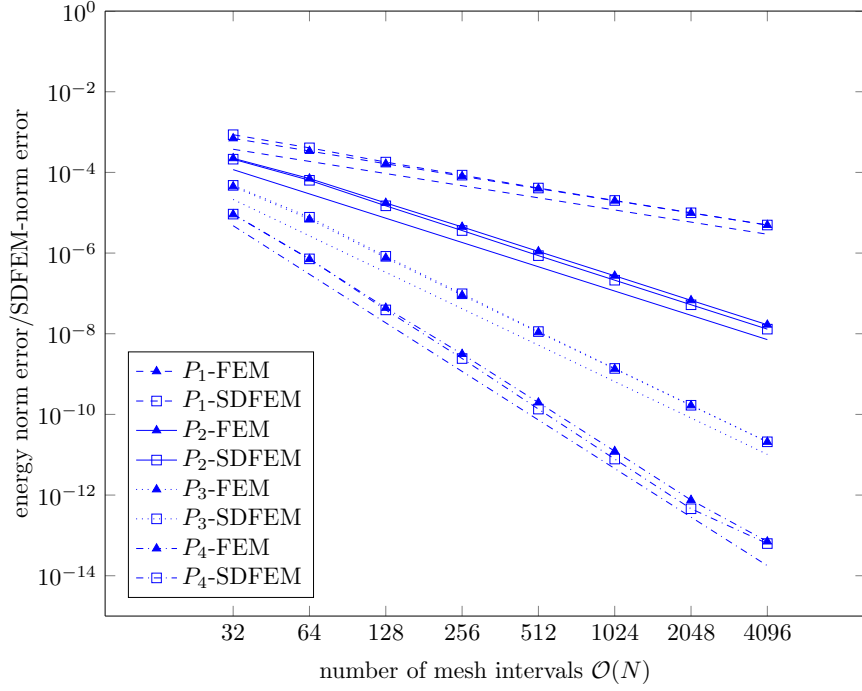


Figure 2: Energy norm error for P_k -FEM and SDFEM-norm error for P_k -SDFEM, $k = 1, \dots, 4$ applied to Example 5.1 with $\varepsilon = 10^{-10}$ and $\lambda = 0.005$. Reference curves of the form $\mathcal{O}(N^{-k})$.

$\varepsilon \setminus N$	P_1 -elements		P_2 -elements		P_3 -elements		P_4 -elements	
	512	1024	512	1024	512	1024	512	1024
1	9.71e-04	4.85e-04	7.49e-07	1.87e-07	9.17e-10	1.15e-10	6.53e-13	2.10e-12
10^{-2}	1.38e-03	6.89e-04	9.60e-06	2.40e-06	4.10e-08	5.12e-09	1.34e-10	8.39e-12
10^{-4}	6.45e-04	3.24e-04	6.73e-06	1.69e-06	4.32e-08	5.44e-09	2.20e-10	1.39e-11
10^{-6}	2.69e-04	1.35e-04	3.76e-06	9.40e-07	3.22e-08	4.02e-09	2.20e-10	1.38e-11
10^{-8}	1.06e-04	5.26e-05	1.90e-06	4.70e-07	1.99e-08	2.45e-09	1.82e-10	1.11e-11
10^{-10}	3.97e-05	1.98e-05	1.10e-06	2.73e-07	1.08e-08	1.34e-09	1.97e-10	1.23e-11
10^{-12}	1.47e-05	7.25e-06	1.06e-06	2.65e-07	5.51e-09	6.69e-10	3.21e-10	2.02e-11
10^{-14}	7.48e-06	2.91e-06	1.33e-06	3.34e-07	5.05e-09	4.17e-10	5.59e-10	3.51e-11

Table 1: Energy norm error for P_k -FEM, $k = 1, \dots, 4$ applied to Example 5.1 with certain ε and $\lambda = 0.005$.

$\varepsilon \setminus N$	P_1 -elements		P_2 -elements		P_3 -elements		P_4 -elements	
	512	1024	512	1024	512	1024	512	1024
1	9.71e-04	4.85e-04	7.49e-07	1.87e-07	9.17e-10	1.15e-10	7.11e-13	2.43e-12
10^{-2}	1.39e-03	6.90e-04	1.02e-05	2.44e-06	4.98e-08	5.39e-09	2.26e-10	9.54e-12
10^{-4}	6.46e-04	3.24e-04	6.76e-06	1.70e-06	4.40e-08	5.56e-09	2.68e-10	1.90e-11
10^{-6}	2.69e-04	1.35e-04	3.76e-06	9.39e-07	3.22e-08	4.02e-09	2.19e-10	1.37e-11
10^{-8}	1.06e-04	5.27e-05	1.85e-06	4.58e-07	2.00e-08	2.46e-09	1.73e-10	1.05e-11
10^{-10}	4.10e-05	2.02e-05	8.64e-07	2.13e-07	1.13e-08	1.38e-09	1.35e-10	7.82e-12
10^{-12}	1.95e-05	8.58e-06	4.89e-07	1.09e-07	7.98e-09	8.53e-10	1.70e-10	8.37e-12
10^{-14}	1.72e-05	6.28e-06	5.01e-07	9.67e-08	1.02e-08	9.24e-10	3.02e-10	1.38e-11

Table 2: SDFEM-norm error for P_k -SDFEM, $k = 1, \dots, 4$ applied to Example 5.1 with certain ε and $\lambda = 0.005$.

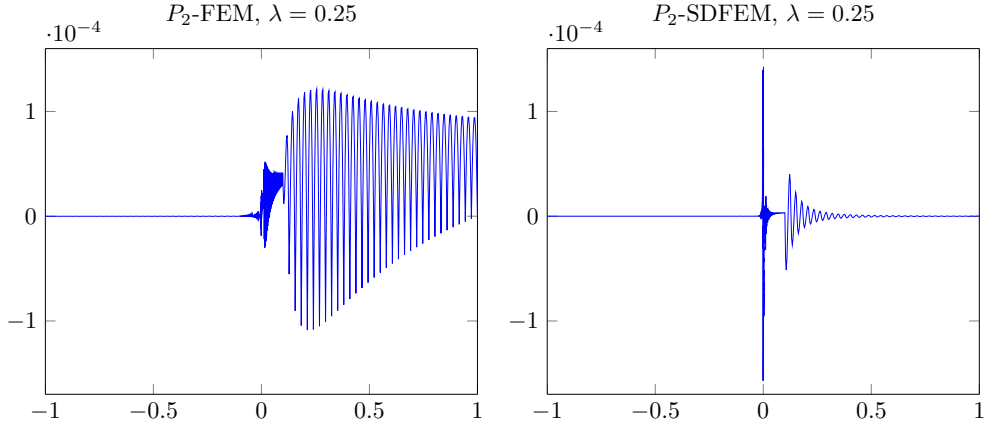


Figure 3: Error for P_2 -FEM (left) and P_2 -SDFEM (right) applied to Example 5.1 with $\varepsilon = 10^{-6}$ and $\lambda = 0.25$ on a mesh with $N = 128$.

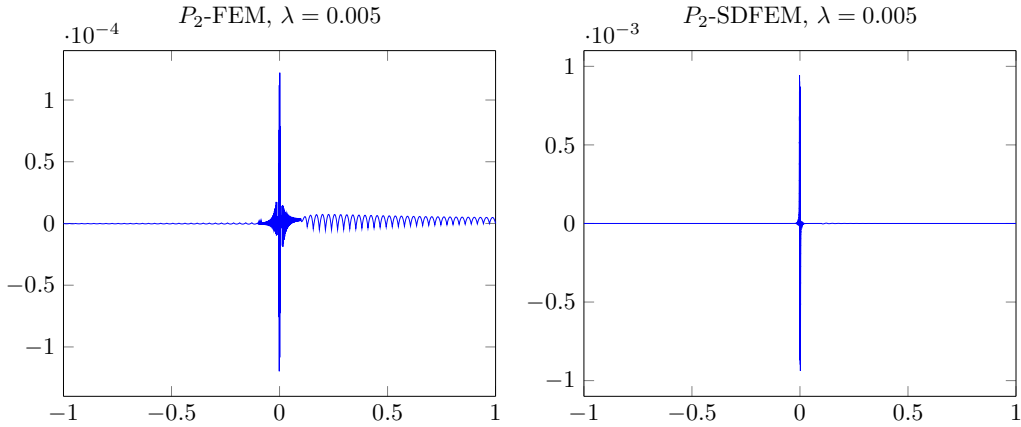


Figure 4: Error for P_2 -FEM (left) and P_2 -SDFEM (right) applied to Example 5.1 with $\varepsilon = 10^{-6}$ and $\lambda = 0.005$ on a mesh with $N = 128$.

N	$ u - u_N _{\text{SD}}$		$ u - u_N _{\varepsilon}$		$ u - u_N $	
	error	rates	error	rates	error	rates
8	2.23e-02	2.584	2.16e-02	2.601	2.16e-02	3.177
16	3.71e-03	2.110	3.57e-03	2.493	2.39e-03	2.989
32	8.60e-04	1.081	6.33e-04	0.943	3.01e-04	1.699
64	4.07e-04	1.174	3.30e-04	1.045	9.27e-05	1.982
128	1.80e-04	1.075	1.60e-04	0.995	2.35e-05	2.006
256	8.56e-05	1.062	8.02e-05	1.016	5.84e-06	2.086
512	4.10e-05	1.023	3.96e-05	1.000	1.38e-06	2.059
1024	2.02e-05	1.016	1.98e-05	1.004	3.30e-07	2.055
2048	9.97e-06	1.006	9.88e-06	1.000	7.95e-08	2.032

Table 3: SDFEM-norm, energy norm, and L^2 -norm error for linear SDFEM applied to Example 5.1 with $\varepsilon = 10^{-10}$ and $\lambda = 0.005$.

we have first order convergence in the SDFEM-norm and second order convergence in the L^2 -norm. But recall that in Section 4.2 we were able to prove the convergence rate $3/2$ only.

Furthermore, we want to present some numerical computations for $\lambda = 0.25$. In Table 4 and Table 5 the energy and L^2 -norm errors are given together with the associated convergence rates for certain ε and P_k -FEM, $k = 1, \dots, 4$. In the studied range of N the results qualitatively differ from the results for smaller λ (before $\lambda = 0.005$ was studied). On the one hand for $\varepsilon \geq 10^{-4}$ the expected convergence behaviour can be seen. Otherwise for smaller ε the L^2 -norm part dominates the energy norm error which may suggest that this norm is too weak. Here we also have to differentiate between odd and even element orders. For $k = 1, 3$ we obtain the convergence order $k + 1$ in the L^2 -norm. So for small ε also the rate in the $|||\cdot|||_{\varepsilon}$ -norm is calculated to be $k + 1$ which surpasses the usual expectations. Otherwise for $k = 2, 4$ we see the expected rate k in the energy norm and when $\varepsilon \leq 10^{-8}$ the same rate also for $||u - u_N||$ where one would rather expect $k + 1$ as obtained for larger ε .

Finally, we want to point out that the detailed structure of the error estimate becomes visible for $\lambda = 0.25$ at least for P_2 -elements. In order to check this we additionally calculate the ratio $|||u - u_N|||_{\varepsilon} \cdot 100(N/(\mathcal{K} + 1))^2$, see Table 6. Excluding the results for $\varepsilon \geq 10^{-2}$ the ratio is nearly independent of ε . Especially in the lower right corner (for $N \geq 256$ and $\varepsilon \leq 10^{-10}$) the ratio is even nearly constant.

In summary, our numerical experiments confirm the theoretical results of Section 3 and Section 4. However, the calculations suggest that the given bound for the L^2 -norm error is not optimal yet. Also some interesting effects differing between odd and even order elements could be seen when λ is not that small.

Acknowledgement

The author would like to thank Hans-Görg Roos and Sebastian Franz for helpful comments and discussions.

$\varepsilon \setminus N$	P_1 -elements				P_2 -elements			
	$ u - u_N _\varepsilon$		$\ u - u_N\ $		$ u - u_N _\varepsilon$		$\ u - u_N\ $	
	512	1024	512	1024	512	1024	512	1024
1	8.06e-04	4.03e-04	8.65e-07	2.16e-07	5.94e-07	1.49e-07	3.22e-10	4.03e-11
	1.000		2.000		2.000		3.000	
10^{-2}	7.80e-04	3.90e-04	5.63e-06	1.41e-06	4.39e-06	1.10e-06	2.38e-08	2.97e-09
	1.000		2.000		2.000		3.000	
10^{-4}	2.18e-04	1.09e-04	5.39e-06	1.30e-06	1.93e-06	4.70e-07	6.27e-08	6.74e-09
	0.998		2.052		2.037		3.218	
10^{-6}	5.63e-05	2.67e-05	1.89e-05	4.03e-06	4.43e-06	1.01e-06	3.46e-06	4.85e-07
	1.076		2.225		2.139		2.834	
10^{-8}	3.28e-05	9.85e-06	3.05e-05	7.87e-06	7.61e-06	1.83e-06	7.57e-06	1.80e-06
	1.735		1.956		2.055		2.069	
10^{-10}	4.84e-05	1.11e-05	4.83e-05	1.11e-05	1.14e-05	2.86e-06	1.14e-05	2.85e-06
	2.121		2.128		1.992		1.992	
10^{-12}	5.58e-05	1.48e-05	5.58e-05	1.48e-05	1.55e-05	3.91e-06	1.55e-05	3.91e-06
	1.915		1.916		1.986		1.986	
10^{-14}	8.88e-05	2.23e-05	8.88e-05	2.23e-05	2.06e-05	5.16e-06	2.06e-05	5.16e-06
	1.992		1.992		1.998		1.998	

Table 4: Energy norm and L^2 -norm error for P_k -FEM, $k = 1, 2$ applied to Example 5.1 with certain ε and $\lambda = 0.25$. Associated convergence rates.

$\varepsilon \setminus N$	P_3 -elements				P_4 -elements			
	$ u - u_N _\varepsilon$		$\ u - u_N\ $		$ u - u_N _\varepsilon$		$\ u - u_N\ $	
	512	1024	512	1024	512	1024	512	1024
1	6.72e-10	8.40e-11	3.92e-13	2.51e-13	7.77e-12	2.83e-11	3.70e-12	1.36e-11
	3.000		0.643		-1.868		-1.875	
10^{-2}	1.66e-08	2.08e-09	6.15e-11	4.59e-12	6.70e-11	4.24e-12	1.93e-13	3.93e-13
	3.000		3.745		3.982		-1.028	
10^{-4}	1.21e-08	1.49e-09	3.24e-10	2.17e-11	9.80e-11	6.02e-12	3.08e-12	1.17e-13
	3.013		3.901		4.025		4.726	
10^{-6}	6.36e-09	6.98e-10	1.83e-09	6.78e-11	3.82e-10	1.93e-11	1.51e-10	3.47e-12
	3.188		4.754		4.309		5.443	
10^{-8}	7.19e-09	5.29e-10	6.80e-09	4.57e-10	9.38e-10	5.72e-11	9.18e-10	5.46e-11
	3.764		3.897		4.036		4.073	
10^{-10}	1.68e-08	8.55e-10	1.67e-08	8.50e-10	2.10e-09	1.33e-10	2.10e-09	1.33e-10
	4.293		4.299		3.978		3.979	
10^{-12}	2.39e-08	1.52e-09	2.39e-08	1.52e-09	3.91e-09	2.50e-10	3.91e-09	2.50e-10
	3.973		3.973		3.969		3.969	
10^{-14}	5.56e-08	3.56e-09	5.56e-08	3.56e-09	6.93e-09	4.36e-10	6.93e-09	4.36e-10
	3.967		3.967		3.991		3.991	

Table 5: Energy norm and L^2 -norm error for P_k -FEM, $k = 3, 4$ applied to Example 5.1 with certain ε and $\lambda = 0.25$. Associated convergence rates.

$\varepsilon \setminus N$	8	16	32	64	128	256	512	1024	2048	4096
1	3.82	3.88	3.89	3.89	3.90	3.90	3.90	3.90	3.90	3.90
10^{-1}	8.99	10.05	10.72	10.88	10.92	10.93	10.94	10.94	10.94	10.94
10^{-2}	20.95	26.38	28.34	28.68	28.74	28.75	28.76	28.76	28.76	28.76
10^{-3}	5.58	7.49	9.35	8.66	8.47	8.28	8.31	8.30	8.31	8.31
10^{-4}	5.15	7.06	8.24	8.50	7.40	6.21	5.61	5.47	5.41	5.41
10^{-5}	5.10	6.64	7.23	7.33	7.39	7.08	5.52	3.48	2.27	1.78
10^{-6}	5.11	6.70	7.64	7.78	7.46	7.33	7.26	6.59	4.60	2.69
10^{-7}	3.28	4.74	6.38	7.76	7.84	7.64	7.33	7.26	7.18	6.49
10^{-8}	3.29	4.75	6.44	7.87	7.98	8.01	7.98	7.69	7.32	7.24
10^{-9}	2.34	4.85	5.95	7.58	7.83	8.22	8.17	8.19	8.03	7.66
10^{-10}	2.34	4.86	5.95	7.60	7.85	8.27	8.27	8.32	8.24	8.18
10^{-11}	2.34	3.63	6.02	6.94	7.72	8.21	8.27	8.34	8.38	8.32
10^{-12}	2.34	3.63	6.02	6.94	7.72	8.22	8.28	8.36	8.41	8.40
10^{-13}	2.34	3.63	6.02	6.94	7.72	8.22	8.28	8.37	8.42	8.42
10^{-14}	2.34	5.10	6.78	7.83	8.27	8.40	8.44	8.45	8.45	8.45

Table 6: $\|u - u_N\|_\varepsilon \cdot 100(N/(\mathcal{K} + 1))^2$ for P_2 -FEM applied to Example 5.1 with certain ε and $\lambda = 0.25$.

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